Vectorial Representation of Uniform Circular Motion

Claim: $\vec{r}(t) = A\cos(\omega t+\phi)\hat{i} + A\sin(\omega t+\phi)\hat{j}$ is the position vector of a particle moving about the origin (i) in a circle of radius A, (ii) starting at an angle ϕ, (iii) with CCW angular velocity $\omega = \frac{2\pi}{T}$.

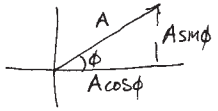
Proof: (i) $|\vec{r}| = \sqrt{r_x^2 + r_y^2} = \sqrt{A^2\cos^2(\omega t+\phi) + A^2\sin^2(\omega t+\phi)}$

$= \sqrt{A^2[\cos^2(\omega t+\phi) + \sin^2(\omega t+\phi)]} = \sqrt{A^2} = A.$

$\therefore$ the particle is always a distance A from the origin.

(ii) $\vec{r}(0) = A\cos\phi\,\hat{i} + A\sin\phi\,\hat{j}$.

$\therefore$ the particle is at (A,ϕ) at $t=0$



(iii) When t increases by T:

$\vec{r}(t+T) = A\cos[\omega(t+T)+\phi]\hat{i} + A\sin[\omega(t+T)+\phi]\hat{j}$

$= A\cos(\omega t+\phi+\omega T)\hat{i} + A\sin(\omega t+\phi+\omega T)\hat{j}.$

But, by definition of T, $\vec{r}(t+T) = \vec{r}(t)$. (Particle is just back to starting pt. after one period T.)

$\Rightarrow A\cos(\omega t+\phi+\omega T)\hat{i} + A\sin(\omega t+\phi+\omega T) = A\cos(\omega t+\phi)\hat{i} + A\sin(\omega t+\phi)\hat{j}$

$\Rightarrow \cos(\omega t+\phi+\omega T) = \cos(\omega t+\phi)$, and

$\sin(\omega t+\phi+\omega T) = \sin(\omega t+\phi+\omega T)$

$\Rightarrow \omega T = 2\pi \Rightarrow \omega = \frac{2\pi}{T}$

Note: for CW circular motion $\omega = -\frac{2\pi}{T}$

Useful relations:

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Problems:

1. Write the position vector of a particle moving in a circle
   a) CCW, starting @ (0,2), with period 1s.
   b) CW, starting @ (2,0), with period 1s.
   c) CW, starting @ (1,1) " " "
   d) CCW, " @ $(\sqrt{3},1)$ " " "

   Reduce to simplest form and prove that $|\vec{r}|$ = the initial distance from the origin.

2. For each of the following circular motions, identify the radius A, period T, phase ϕ, and direction of rotation.
   a) $\sqrt{5}\cos t\,\hat{i} + \sqrt{5}\sin t\,\hat{j}$
   b) $0.1\sin\pi t\,\hat{i} - 0.1\cos\pi t\,\hat{j}$
   c) $-0.1\cos\pi^2 t\,\hat{i} + 0.1\sin\pi^2 t\,\hat{j}$

Make up your own problems!

Answers:

1. a) $-2\sin 2\pi t\,\hat{i} + 2\cos 2\pi t\,\hat{j}$
   b) $2\cos 2\pi t\,\hat{i} - 2\sin 2\pi t\,\hat{j}$
   c) $\sqrt{2}\left(\cos 2\pi t - \frac{\pi}{4}\right)\hat{i} - \sqrt{2}\sin\left(2\pi t - \frac{\pi}{4}\right)\hat{j}$
   d) $2\cos\left(2\pi t + \frac{\pi}{6}\right)\hat{i} + 2\sin\left(2\pi t + \frac{\pi}{6}\right)\hat{j}$

2. a) radius $\sqrt{5}$, period 2π s, $\phi = 0$, CCW.
   b) radius 0.1, period 2s, $\phi = -90°$, CCW.
   c) radius 0.1, period $\frac{2}{\pi}$ s, $\phi = 180°$, CW.